Spatial and null infinity via advanced and retarded conformal factors

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A new approach to space-time asymptotics is presented, refining Penrose's idea of conformal transformations with infinity represented by the conformal boundary of space-time. Generalizing examples such as flat and Schwarzschild space-times, it is proposed that the Penrose conformal factor be a product of *advanced and retarded conformal factors*, which asymptotically relate physical and conformal null (light-like) coordinates and vanish at future and past null infinity respectively, with both vanishing at spatial infinity. A correspondingly refined definition of asymptotic flatness at both spatial and null infinity is given, including that the conformal boundary is locally a light cone, with spatial infinity as the vertex. It is shown how to choose the conformal factors so that this asymptotic light cone is locally a metric light cone. The theory is implemented in the spin-coefficient (or null-tetrad) formalism by a simple joint transformation of the spin-metric and spin-basis (or metric and tetrad). The advanced and retarded conformal factors may be used as expansion parameters near the respective null infinity, together with a dependent expansion parameter for both spatial and null infinity, essentially inverse radius. Asymptotic regularity conditions on the spin-coefficients are proposed, based on the conformal boundary locally being a smoothly embedded metric light cone. These conditions ensure that the Bondi-Sachs energy-flux integrals of ingoing and outgoing gravitational radiation decay at spatial infinity such that the total radiated energy is finite, and that the Bondi-Sachs energy-momentum has a unique limit at spatial infinity, coinciding with the uniquely rendered ADM energy-momentum.

I. INTRODUCTION

Space-time asymptotics, the study of isolated gravitational systems at large distances, is one of the theoretical pillars of General Relativity, whereby one can define physically important quantities which are difficult to capture in general, such as gravitational radiation, the energy flux of gravitational radiation and the active gravitational mass-energy of the system. The discovery of the Bondi-Sachs energy-loss equation [1, 2, 3], relating the change in total mass-energy to the energy flux of outgoing gravitational radiation, just as for electromagnetic radiation, marked a transition from an epoch where some argued that Einstein's original prediction of gravitational radiation rested on mathematical artefacts, to the present epoch where there is significant investment in expectations of detecting gravitational radiation from astrophysical sources.

The early work on space-time asymptotics was brilliantly reformulated by Penrose [4, 5], who realized how to handle infinite distances and times in a mathematically finite way. The space-time metric is mapped to a multiple of itself, the factor tending to zero in such a way that the new metric is finite and well behaved at the physical space-time infinity. The mapping is described as conformal since it preserves space-time angles and therefore causal relations, while changing distances and durations. In the conformal picture, infinity becomes a finite boundary where one can derive exact formulas, encapsulating physical laws which were otherwise approximate or limiting.

Penrose's conformal theory rapidly became the standard framework for space-time asymptotics at null (light-like) infinity, and was quite thoroughly implemented using the spin-coefficient or null-tetrad formalism [5, 6, 7, 8, 9]. However, it does not cover spatial infinity, described initially by the ADM method [10] and by a spatial version of Penrose's method by Geroch [11]. The essential problem is to describe spatial and null infinity in a unified way. Ashtekar and coworkers [12, 13, 14] proposed and investigated such a definition, realizing that spatial infinity is generally a directional singularity of the conformal metric. However, there still seems to be no practical calculational formalism. Some physically important issues are whether the gravitational radiation decays near spatial infinity such that the Bondi-Sachs energy exists and coincides with the ADM energy at spatial infinity, and whether initial data on an asymptotically flat spatial hypersurface (or past null infinity) determines final data at future null infinity. For a more recent perspective, see e.g. Friedrich [15].

This article presents a quite simple, natural unification of spatial and null infinity, based on a re-examination of Penrose's original insights. The key new idea is that Penrose's conformal factor should be a product of advanced and retarded conformal factors, which do the work of the conformal factor at future and past null infinity respectively, while cooperating at spatial infinity so that it is the vertex of a light cone, generating null infinity. These factors, squared, differentially relate physical and conformal null coordinates near the respective null infinity, up to the relevant coordinate freedom.

The basic idea is most easily apprehended in examples, described in §II. §III gives a definition of asymptotic flatness intended to encapsulate the new refinements, and shows that conformal infinity is locally a metric light cone, thereby determining standard coordinates. §IV shows how to implement the theory in the spin-coefficient (or null-tetrad) formalism, using a simple joint transformation of the spin-metric and spin-basis (or metric and null tetrad). §V identifies appropriate expansion parameters and proposes asymptotic regularity conditions at both null and spatial infinity, based on the asymptotic light cone being smoothly embedded. §VI uses the Hawking quasi-local energy [16] to study the Bondi-Sachs energy flux (of ingoing and outgoing gravitational radiation) and total energy at both null and spatial infinity, and the ADM energy. §VII concludes.

II. ADVANCED AND RETARDED CONFORMAL FACTORS

For flat space-time, the line element may be written as

$$ds^2 = r^2 dS^2 + dr^2 - dt^2 \qquad (1)$$

where dS^2 is a line element for the unit sphere. In terms of dual-null (or characteristic) coordinates

$$2\xi^\pm = t \pm r \qquad (2)$$

it becomes

$$ds^2 = (\xi^+ - \xi^-)^2 dS^2 - 4d\xi^+ d\xi^-. \qquad (3)$$

To approach infinity, the physical null coordinates $\xi^\pm$ are transformed to conformal null coordinates $\psi^\pm$ by

$$\xi^\pm = \tan \psi^\pm. \qquad (4)$$

This describes sterographic projection, used to map the Earth on flat paper: if one projects from the north pole of a unit sphere onto the equatorial plane, with ψ the angle from the vertical, then $\xi = \tan\psi$ is the distance from the centre of a point in the plane. Thus the infinite plane is mapped into the compact sphere, with the north pole representing infinity; the sphere describes the whole plane, plus its infinity. The formula also occurs in complex analysis as a way of using the Riemann sphere to represent the Argand plane, plus a point at infinity.

The novel point concerns the derivatives occurring in the null coordinate transformations,

$$(\omega^\pm)^2 = \frac{d\psi^\pm}{d\xi^\pm} = \cos^2 \psi^\pm \qquad (5)$$

which vanish at the relevant infinity. The line element written in $\psi^\pm$ coordinates therefore contains a term $4d\psi^+ d\psi^- / (\omega^+ \omega^-)^2$. To obtain a metric which is regular at infinity, one can multiply by the discrepancy, the square of

$$\Omega = \omega^+ \omega^-. \qquad (6)$$

Then

$$\Omega^2 ds^2 = \sin^2(\psi^+ - \psi^-)dS^2 - 4d\psi^+ d\psi^-. \qquad (7)$$

A final transformation

$$2\psi^\pm = \tau \mp \rho \qquad (8)$$

puts this in the form

$$\Omega^2 ds^2 = \sin^2 \rho \, dS^2 + d\rho^2 - d\tau^2 \qquad (9)$$

which is the standard line element of the Einstein static universe, a metric $S^3 \times R$. The physical space-time metric g has been mapped to the conformal metric $\Omega^2 g$ and the physical space-time manifold has been mapped to the conformal coordinate range $\psi^\pm \in (-\pi/2, \pi/2)$. Thus physical infinity has been mapped to the boundary of this region. This conformal boundary consists of two null hypersurfaces connected by three points: future null infinity $\Im^+$, $\psi^+ = \pi/2$, $\psi^- \in (-\pi/2, \pi/2)$; past null infinity $\Im^-$, $\psi^- = -\pi/2$, $\psi^+ \in (-\pi/2, \pi/2)$; future temporal infinity i^+, $\psi^+ = \psi^- = \pi/2$; past temporal infinity i^-, $\psi^+ = \psi^- = -\pi/2$; and spatial infinity i^0, $\psi^+ = -\psi^- = \pm\pi/2$. The two null hypersurfaces locally have the structure of light cones, with the three points each being vertices. This initially bizarre sixties spacewarp led Penrose [4, 5] to the definition of conformal infinity as essentially $\Omega = 0$, $\nabla\Omega \neq 0$, the physical metric g admitting a well behaved conformal metric $\Omega^2 g$, Ω being called the conformal factor.

Nevertheless, there is more structure here. Firstly, spatial infinity is the vertex of a conformal light cone consisting of null infinity. Secondly, $\Im^\pm$ are given by $\omega^\pm = 0$, $\omega^\mp \neq 0$, with $\omega^+ = \omega^- = 0$ at spatial infinity. Clearly there is more information in the sub-factors $\omega^\pm$ than in the Penrose factor Ω alone. This structure will be used to refine Penrose's definition of asymptotic flatness. Henceforth ω^+ and ω^- will be called the *advanced and retarded conformal factors* respectively.

As a second example, the Schwarzschild space-time with mass M in standard coordinates is

$$ds^2 = r^2 dS^2 + (1 - 2M/r)^{-1} dr^2 - (1 - 2M/r)dt^2. \qquad (10)$$

This can be written in dual-null coordinates

$$2\xi^\pm = t \pm r_* \qquad (11)$$

where

$$r_* = r + 2M \ln(r - 2M) \qquad (12)$$

rescales r such that $dr/dr_* = 1 - 2M/r$. Then

$$ds^2 = r^2 dS^2 - (1 - 2M/r)4d\xi^+ d\xi^- \qquad (13)$$

where r is implicitly determined as a function of $r_* = \xi^+ - \xi^-$. As before, the physical null coordinates $\xi^\pm$ may be transformed to conformal null coordinates $\psi^\pm$ by, for instance, simple inversion:

$$\xi^\pm = -1/\psi^\pm. \qquad (14)$$

This is valid only in a neighbourhood of spatial infinity, but is simple enough to constitute a local canonical transformation. The derivatives occurring in the null coordinate transformations are

$$(\omega^{\pm})^2 = \frac{d\psi^{\pm}}{d\xi^{\pm}} = (\psi^{\pm})^2 \qquad (15)$$

and one can fix $\omega^{\pm} = \mp\psi^{\pm}$ so that $\omega^{\pm} > 0$ in the physical space-time, $\xi^{\pm}$ and $\psi^{\pm}$ being future-pointing. As before, the metric in $\psi^{\pm}$ coordinates can be regularized by multiplying by the square of $\Omega = \omega^+\omega^-$. Then

$$\Omega^2 ds^2 = (\psi^+\psi^- r)^2 dS^2 - 4(1 - 2M/r)d\psi^+ d\psi^-. \quad (16)$$

Transforming as before by (8) puts this in the form

$$\Omega^2 ds^2 = (\rho r/r_*)^2 dS^2 + (1 - 2M/r)(d\rho^2 - d\tau^2). \quad (17)$$

Since $r_*/r \to 1$ as $r \to \infty$, the conformal metric becomes flat at infinity, with ρ playing the role of conformal radius:

$$\Omega^2 ds^2 \to \rho^2 dS^2 + d\rho^2 - d\tau^2. \qquad (18)$$

Thus infinity consists of a locally metric light cone with spatial infinity as the vertex.

Space-time has been turned inside out: spatial infinity has become a point and the physical space-time lies outside its light cone, as depicted in Fig. 1. In more detail, the original region $\xi^+ > 0$, $\xi^- < 0$ has been mapped to $\psi^+ < 0$, $\psi^- > 0$, or $\omega^{\pm} > 0$. The conformal boundary again locally consists of $\Im^{\pm}$, where $\omega^{\pm} = 0$, $\omega^{\mp} \neq 0$, and spatial infinity i^0, where $\omega^+ = \omega^- = 0$. The case $M = 0$ was not excluded, so this also provides a conformal transformation of flat space-time such that infinity is locally a metric light cone at spatial infinity. The key points are the identification of the advanced and retarded conformal factors $\omega^{\pm}$ as differentially relating physical and conformal null coordinates, and their behaviour at null and spatial infinity.

III. THE LIGHT CONE AT INFINITY

The above insights are now converted into a definition of asymptotic flatness at both null and spatial infinity. In this section alone, g will be used for the conformal metric and $\tilde{g}$ for the physical metric. Quantities without tildes, e.g. ∇, similarly refer to the conformal metric.

In this article, a space-time $(\tilde{M}, \tilde{g})$ will be said to be asymptotically flat if the following conditions hold. (i) There exists a space-time (M, g) with boundary $\Im = \partial M$ and functions $\omega^{\pm} > 0$ on $\tilde{M}$ such that $M = \tilde{M} \cup \Im$ and $g = (\omega^+\omega^-)^2\tilde{g}$ in $\tilde{M}$. (ii) $\Im$ is locally a light cone with respect to g, with vertex denoted by i^0 and future and past open cones denoted by $\Im^+$ and $\Im^-$ respectively. (iii) $\omega^{\pm} = 0$, $\omega^{\mp} \neq 0$, $\nabla\omega^{\pm} \neq 0$ on $\Im^{\pm}$, and $g^{-1}(\nabla\omega^+, \nabla\omega^-) \neq 0$ on $\Im$. (iv) $(g, \omega^{\pm})$ are C^k on $M - i^0$, with $k = 3$ sufficing for the purposes of this article. Differentiability at i^0 is deferred to §V.
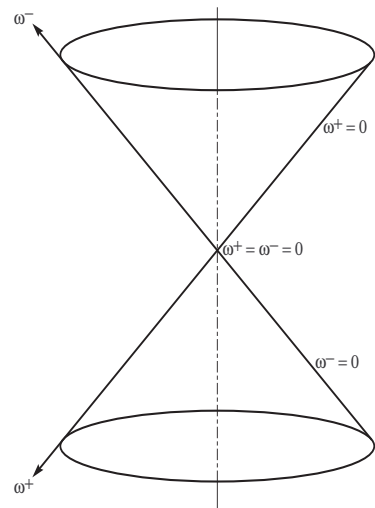

FIG. 1: The light cone at infinity, depicting the advanced and retarded conformal factors $\omega^{\pm}$. The physical space-time ($\omega^+ > 0$, $\omega^- > 0$) lies outside this asymptotic light cone. Radiation propagates in from past null infinity $\Im^-$ ($\omega^- = 0$) and out to future null infinity $\Im^+$ ($\omega^+ = 0$). Spatial infinity i^0 ($\omega^+ = \omega^- = 0$) is the vertex. For non-zero total mass, beware the singularity beyond infinity.

One may call $\Im = \Im^+ \cup \Im^- \cup i^0$ (locally) the asymptotic light cone, or more prosaically, the *light cone at infinity* [12]. The definition can be easily seen to recover the basic conditions of Penrose's definition of asymptotic simplicity [4, 5], namely $\Omega = 0$ and $\nabla\Omega \neq 0$ on $\Im^{\pm}$, with the Penrose factor Ω as before (6). The causal restriction on (plain or weak) asymptotic simplicity has been replaced with the light-cone condition. Note by continuity that $\omega^+ = \omega^- = 0$ at i^0. The differentiability conditions at spatial infinity are notoriously awkward: if g is C^1 then the total (ADM) energy vanishes, while if g is only C^0, the total energy is generally ill-defined [12, 13, 14]. Readers unable to proceed otherwise can provisionally assume the Ashtekar-Hansen differentiability conditions at i^0. Essentially, a non-zero mass forces radial derivatives of the conformal metric to be discontinuous at spatial infinity, even for the Schwarzschild metric, while angular derivatives should be much smoother. A geometrical way to understand this stems from the fact that the positive-mass Schwarzschild space-time can be conformally extended through $\Im^{\pm}$ to smoothly include a negative-mass Schwarzschild space-time whose ($\Im^{\mp}, i^{\mp}$) coincide with the physical ($\Im^{\pm}, i^0$) [5]. Since the negative-mass Schwarzschild space-time has a naked curvature singularity extending from its i^- to i^+, this *singularity beyond infinity* (Fig. 1) piercing physical i^0 can be understood as forcing spatial infinity to be a directional singularity of the conformal metric.

The analysis of Geroch [11] for $\Im^{\pm}$, summarized together with the Ashtekar-Hansen analysis by Wald [17], can be taken as a guide to the following refinements.

Firstly, the definition allows considerable gauge freedom in the conformal factors $\omega^\pm$, namely $\omega^\pm \mapsto \alpha_\pm \omega^\pm$ for any C^k functions $\alpha_\pm > 0$ on $M - i^0$. This gauge freedom allows the conformal metric of $\Im$ to be locally fixed as that of a metric cone, as follows. The starting point is a standard expression for the physical Ricci tensor $\tilde{R}$ in terms of the conformal Ricci tensor R:

$$\Omega\tilde{R} = \Omega R + 2\nabla \otimes \nabla\Omega + (\nabla^2\Omega - 3\Omega^{-1}g^{-1}(\nabla\Omega, \nabla\Omega))g. \tag{19}$$

Applying either the vacuum Einstein equation $\tilde{R} = 0$ or the full Einstein equation with suitable fall-off of the matter fields, one sees that $\Omega^{-1}g^{-1}(\nabla\Omega, \nabla\Omega)$ is smooth (C^{k-2}) at $\Im^\pm$ and therefore that $\nabla\Omega$ is null there. Further, expanding

$$\frac{g^{-1}(\nabla\Omega, \nabla\Omega)}{\Omega} = 2g^{-1}(\nabla\omega^+, \nabla\omega^-) + \tag{20}$$

$$\frac{\omega^-}{\omega^+}g^{-1}(\nabla\omega^+, \nabla\omega^+) + \frac{\omega^+}{\omega^-}g^{-1}(\nabla\omega^-, \nabla\omega^-)$$

and using the above definition, one finds that $(\omega^\mp/\omega^\pm)g^{-1}(\nabla\omega^\pm, \nabla\omega^\pm)$ are smooth at $\Im^\pm$ and therefore that the 1-forms

$$n^\pm = \pm 2\nabla\omega^\pm \tag{21}$$

are null at $\Im^\pm$, where the signs are chosen so that the vectors $g^{-1}(n^\pm)$ are future pointing, taking the choice $\omega^\pm > 0$ in $\tilde{M}$ and the $(-+++)$ metric convention. The gauge freedom $\omega^\pm \mapsto \alpha_\pm\omega^\pm$ can then be used to set $g^{-1}(\nabla\omega^\pm, \nabla\omega^\pm)/\omega^\pm = 0$ on $\Im^\pm$, by an argument given by Geroch for Ω. This still leaves $\omega^\mp$ free on $\Im^\pm$. The normalization condition, which now reads $g^{-1}(\nabla\omega^+, \nabla\omega^-) > 0$ on $\Im$, expresses that the hypersurfaces of constant $\omega^\pm$ intersect $\Im^\pm$ transversely; then the gauge freedom allows one to choose them to intersect in a null direction, $g^{-1}(\nabla\omega^\mp, \nabla\omega^\mp) = 0$ on $\Im^\pm$. Thus we have two independent null directions $g^{-1}(n^\pm)$ at $\Im^+$ and $\Im^-$. The local null rescaling freedom of a null hypersurface can then be used to further adjust $\omega^\mp$ on $\Im^\pm$ so that the null directions are relatively normalized: $g^{-1}(\nabla\omega^+, \nabla\omega^-) = 1/2$ on $\Im^\pm$. All the above conditions extend to i^0 by continuity. Using $\approx$ to denote equality on $\Im$ in a neighbourhood of i^0, the gauge conditions are

$$g^{-1}(\nabla\omega^\pm, \nabla\omega^\pm)/\omega^\pm \approx 0 \tag{22}$$
$$g^{-1}(\nabla\omega^+, \nabla\omega^-) \approx 1/2. \tag{23}$$

Together they imply that

$$g^{-1}(\nabla\Omega, \nabla\Omega)/\Omega \approx 1 \tag{24}$$

which shows that Ω is spatial in a neighbourhood of $\Im$. The gauge condition usually chosen for $\Im^+$ is that this quantity vanishes.

Taking the trace and rearranging, the curvature relation (19) then implies

$$2\nabla \otimes \nabla\Omega \approx g \tag{25}$$

which again differs from the usual relation $\nabla \otimes \nabla\Omega = 0$ on $\Im^+$, which would imply that $\Im^+$ be a metric cylinder. Instead it implies that $\Im$ is a metric cone, as follows. First note that there are now preferred spatial sections of $\Im^\pm$, given by constant $\omega^\mp$. Expanding the above relation,

$$g \approx 2\omega^+\nabla \otimes \nabla\omega^- + 2\omega^-\nabla \otimes \nabla\omega^+ + 4\nabla\omega^+ \otimes \nabla\omega^-. \tag{26}$$

The last term gives the part of the metric normal to the spatial sections of $\Im$, while the first two terms are related to the expansions $\theta_\mp$ and shears $\sigma_\mp$ of $\Im^\pm$ by $\perp 2\nabla \otimes n^\pm = \theta_\mp h + \sigma_\mp$ on $\Im^\pm$, where h is the metric of the spatial sections and $\perp$ denotes projection by h. Since

$$g \approx h + 4\nabla\omega^+ \otimes \nabla\omega^- \tag{27}$$

one reads off $\theta_\mp = \pm 2/\omega^\mp$, $\sigma_\mp = 0$ on $\Im^\pm$. However, this describes a metric light cone, as follows. The flat metric

$$ds^2 = \rho^2 dS^2 + d\rho^2 - d\tau^2 \tag{28}$$

in dual-null coordinates (8) is

$$ds^2 = (\psi^- - \psi^+)^2 dS^2 - 4d\psi^+ d\psi^- \tag{29}$$

where the signs of $\psi^\pm$ have been chosen so that they are future-pointing. Then $\sigma_\mp = 0$ and $\theta_\mp = 2\partial_\mp\rho/\rho = 2/\psi^\mp$ on $\Im^\pm$. Since the expansions and shears, including vertex conditions, characterize the intrinsic geometry of the light cone, this identifies the conformal metric with the flat metric at $\Im$ if

$$\psi^\pm \approx \mp\omega^\pm. \tag{30}$$

In general, $\psi^\pm$ can be taken as null coordinates with this gauge choice at $\Im$. In summary, the conformal metric at $\Im$ has been locally fixed as that of a metric light cone.

IV. FORMALISMS

The most developed implementation of conformal infinity involves the spin-coefficient formalism, also due to Penrose and coworkers [6, 8], which provides the most elegant formulation of various issues such as the Sachs "peeling" of the gravitational field near $\Im^+$ [3, 4, 5]. Henceforth this formalism is employed, though an attempt is made to render the description accessible to the uninitiated. Alternatively, the null-tetrad formalism suffices for most purposes.

Henceforth equation numbers in triples will indicate equations in Penrose & Rindler [5]. It is convenient to modify some notation and summarize, as follows. The basic geometrical object is a 2-form ε (2.5.2) acting on complex 2-spinors (spin-vectors), here called the *spin-metric*. It can be regarded as a square root of the metric, which is expressed as a direct product $g = \varepsilon \circ \bar{\varepsilon}$, where the bar denotes the complex conjugate and the exact meaning of the product is given in abstract index

notation (3.1.9). An ordered pair of non-parallel spin-vectors (o, ι) is here called a *spin-basis*. Expressing vectors as spin-vector dyads, this defines a null tetrad, a vector basis (3.1.14):

$$(l, m, \bar{m}, l') = (o\bar{o}, o\bar{\iota}, \iota\bar{o}, \iota\bar{\iota}). \tag{31}$$

Then (l, l') are real null vectors, whereas m is a complex vector encoding transverse spatial vectors. In terms of the complex normalization factor (2.5.46)

$$\chi = \varepsilon(o, \iota) \tag{32}$$

one finds

$$g(l, l') = -g(m, \bar{m}) = \chi\bar{\chi} \tag{33}$$

with the other eight independent (symmetric) contractions of the tetrad vectors vanishing. Here the metric convention has switched to $(+---)$, as is regrettably standard in the spin-coefficient formalism. Some conventional factors of $\sqrt{2}$ are also entailed compared to the previous sections, e.g. physical null coordinates $(\xi, \xi') = \sqrt{2}(\xi^+, \xi^-)$ and conformal null coordinates $(\psi, \psi') = \sqrt{2}(\psi^+, \psi^-)$.

The inverse metric can be written

$$g^{-1} = \frac{2}{\chi\bar{\chi}}(l \otimes l' - m \otimes \bar{m}) \tag{34}$$

where $\otimes$ denotes the symmetric tensor product. The dual basis of 1-forms (4.13.32) will be denoted by

$$(n, w, \bar{w}, n') = g(l', -\bar{m}, -m, l) \tag{35}$$

so that

$$n(l) = w(m) = \bar{w}(\bar{m}) = n'(l') = 1 \tag{36}$$

with the other twelve such contractions vanishing. Then the metric can be written

$$g = 2\chi\bar{\chi}(n \otimes n' - w \otimes \bar{w}). \tag{37}$$

The tetrad covariant derivative operators (4.5.23) are denoted by

$$(D, \delta, \delta', D') = (\nabla_l, \nabla_m, \nabla_{\bar{m}}, \nabla_{l'}). \tag{38}$$

The complex spin-coefficients $(\kappa, \sigma, \rho, \tau, \varepsilon, \beta, \alpha, \gamma)$ and $(\kappa', \sigma', \rho', \tau', \varepsilon', \beta', \alpha', \gamma')$ encode the Ricci rotation coefficients and can be defined by (4.5.26–27) as

$$D(o, \iota) = (\varepsilon o - \kappa\iota, \gamma'\iota - \tau'o) \tag{39}$$
$$\delta(o, \iota) = (\beta o - \sigma\iota, \alpha'\iota - \rho'o) \tag{40}$$
$$\delta'(o, \iota) = (\alpha o - \rho\iota, \beta'\iota - \sigma'o) \tag{41}$$
$$D'(o, \iota) = (\gamma o - \tau\iota, \varepsilon'\iota - \kappa'o). \tag{42}$$

Other notation will be taken as in Penrose & Rindler [5].

It is convenient to choose null coordinates (ξ, ξ') such that $(n, n') = (d\xi, d\xi')$, which implies the dual-null gauge conditions [18]

$$0 = \kappa = \rho - \bar{\rho} = \varepsilon + \bar{\varepsilon} = \tau - \bar{\alpha} - \beta \tag{43}$$
$$0 = \kappa' = \rho' - \bar{\rho}' = \varepsilon' + \bar{\varepsilon}' = \tau' - \bar{\alpha}' - \beta'. \tag{44}$$

It is also convenient to use Penrose's complex stereographic coordinate (1.2.10)

$$\zeta = e^{i\phi}\cot(\theta/2) \tag{45}$$

where (θ, ϕ) are standard spherical polar coordinates. Then $(w, \bar{w}) = (d\bar{\zeta}/P, d\zeta/\bar{P})$, (4.14.29), where (4.14.31)

$$P = \delta\bar{\zeta}. \tag{46}$$

In summary, we have preferred coordinates $(\xi, \zeta, \bar{\zeta}, \xi')$ such that the 1-form basis is

$$(n, w, \bar{w}, n') = (d\xi, d\bar{\zeta}/P, d\zeta/\bar{P}, d\xi'). \tag{47}$$

The above quantities will refer to the physical metric g and corresponding quantities for the conformal metric $\hat{g}$ will be denoted by hats.

The physical coordinates are now to be transformed to conformal coordinates $(\psi, \zeta, \bar{\zeta}, \psi')$, such that the angular coordinates $\hat{\zeta} = \zeta$ are preserved, while the null coordinates transform to $(\hat{\xi}, \hat{\xi}') = (\psi(\xi), \psi'(\xi'))$, with derivatives

$$(\omega^2, \omega'^2) = \left(\frac{d\psi}{d\xi}, \frac{d\psi'}{d\xi'}\right). \tag{48}$$

Taking ξ as an advanced (outgoing) null coordinate and ξ' as a retarded (ingoing) null coordinate, this means that ω and ω' are the advanced and retarded conformal factors respectively, assumed positive in the space-time. For the canonical example of inversion, $(\psi, \psi') = (-2/\xi, -2/\xi')$ with the current convention, yielding $(\omega, \omega') = (-\psi, \psi')/\sqrt{2}$. The Penrose conformal factor is

$$\Omega = \omega\omega' \tag{49}$$

and the physical spin-metric ε is correspondingly transformed to the conformal spin-metric

$$\hat{\varepsilon} = \omega\omega'\varepsilon \tag{50}$$

and the physical metric g to the conformal metric $\hat{g} = \hat{\varepsilon} \circ \hat{\bar{\varepsilon}}$:

$$\hat{g} = (\omega\omega')^2 g. \tag{51}$$

By comparing the explicit forms

$$g = 2|\chi|^2\left(d\xi \otimes d\xi' - \frac{d\zeta \otimes d\bar{\zeta}}{|P|^2}\right) \tag{52}$$

$$\hat{g} = 2|\hat{\chi}|^2\left(d\psi \otimes d\psi' - \frac{d\zeta \otimes d\bar{\zeta}}{|\hat{P}|^2}\right) \tag{53}$$

one reads off

$$\hat{\chi} = \chi \tag{54}$$

$$\hat{P} = P/\omega\omega'. \tag{55}$$

Then the conformal 1-form basis

$$(\hat{n}, \hat{w}, \hat{\bar{w}}, \hat{n}') = (d\psi, d\bar{\zeta}/\hat{P}, d\zeta/\hat{\bar{P}}, d\psi') \tag{56}$$

is related to the physical 1-form basis by

$$(\hat{n}, \hat{w}, \hat{\bar{w}}, \hat{n}') = (\omega^2 n, \omega\omega' w, \omega\omega' \bar{w}, \omega'^2 n'). \tag{57}$$

Since

$$\hat{g}^{-1} = g^{-1}/(\omega\omega')^2 \tag{58}$$

the physical null tetrad

$$(l, m, \bar{m}, l') = g^{-1}(n', -\bar{w}, -w, n) \tag{59}$$

and the conformal null tetrad

$$(\hat{l}, \hat{m}, \hat{\bar{m}}, \hat{l}') = \hat{g}^{-1}(\hat{n}', -\hat{\bar{w}}, -\hat{w}, \hat{n}) \tag{60}$$

are then related by

$$(\hat{l}, \hat{m}, \hat{\bar{m}}, \hat{l}') = \left(\frac{l}{\omega^2}, \frac{m}{\omega\omega'}, \frac{\bar{m}}{\omega\omega'}, \frac{l'}{\omega'^2}\right). \tag{61}$$

Remarkably, this is just the behaviour of the null tetrad under a transformation of spin-basis (o, ι) to

$$(\hat{o}, \hat{\iota}) = (o/\omega, \iota/\omega'). \tag{62}$$

Such transformations $(o, \iota) \mapsto (\lambda o, \lambda' \iota)$, with (λ, λ') generally complex, (4.12.2), are well understood and led to the concept of weighted scalars (4.12.9) and the compacted spin-coefficient formalism [8]. In summary, it has been shown that *the desired conformal transformation is given by a simultaneous transformation of the spin-metric and spin-basis*, (50) and (62). This can be so without the conformal factors (ω, ω') exactly relating physical and conformal null coordinates as above, due to the considerable freedom in the conformal factors.

The physical and conformal tetrad derivative operators are related by

$$(\hat{D}, \hat{\delta}, \hat{\delta}', \hat{D}') = \left(\frac{D}{\omega^2}, \frac{\delta}{\omega\omega'}, \frac{\delta'}{\omega\omega'}, \frac{D'}{\omega'^2}\right). \tag{63}$$

The relations between the weighted spin-coefficients are obtained straightforwardly from (5.6.15) as

$$\hat{\kappa} = \kappa\omega'/\omega^3 \tag{64}$$

$$\hat{\sigma} = \sigma/\omega^2 \tag{65}$$

$$\hat{\rho} = (\rho - D\log\omega\omega')/\omega^2 \tag{66}$$

$$\hat{\tau} = (\tau - \delta\log\omega\omega')/\omega\omega' \tag{67}$$

$$\hat{\tau}' = (\tau' - \delta'\log\omega\omega')/\omega\omega' \tag{68}$$

$$\hat{\rho}' = (\rho' - D'\log\omega\omega')/\omega'^2 \tag{69}$$

$$\hat{\sigma}' = \sigma'/\omega'^2 \tag{70}$$

$$\hat{\kappa}' = \kappa'\omega/\omega'^3. \tag{71}$$

They are much simpler than the expressions (5.6.25) or (5.6.27) used previously to describe conformal transformations, due to the separation of the Penrose conformal factor into advanced and retarded conformal factors. The components (4.11.6)

$$\Psi_0 = \Psi(o, o, o, o)/|\chi|^2 \tag{72}$$

$$\Psi_1 = \Psi(o, o, o, \iota)/|\chi|^2 \tag{73}$$

$$\Psi_2 = \Psi(o, o, \iota, \iota)/|\chi|^2 \tag{74}$$

$$\Psi_3 = \Psi(o, \iota, \iota, \iota)/|\chi|^2 \tag{75}$$

$$\Psi_4 = \Psi(\iota, \iota, \iota, \iota)/|\chi|^2 \tag{76}$$

of the conformally invariant Weyl curvature spinor $\Psi = \hat{\Psi}$ (4.6.35) transform even more simply as

$$\hat{\Psi}_0 = \Psi_0/\omega^4 \tag{77}$$

$$\hat{\Psi}_1 = \Psi_1/\omega^3\omega' \tag{78}$$

$$\hat{\Psi}_2 = \Psi_2/\omega^2\omega'^2 \tag{79}$$

$$\hat{\Psi}_3 = \Psi_3/\omega\omega'^3 \tag{80}$$

$$\hat{\Psi}_4 = \Psi_4/\omega'^4. \tag{81}$$

One can also generalize the concept of *conformal density* (5.6.32) to a quantity η which transforms as

$$\hat{\eta} = \eta/\omega^k \omega'^{k'} \tag{82}$$

where $\{k, k'\}$ is the *conformal weight*, with signs chosen for convenience. Then the shears (σ, σ') have conformal weights $\{2, 0\}$ and $\{0, 2\}$ respectively. Also $\tau - \bar{\tau}'$ has conformal weight $\{1, 1\}$. Generally $\tau + \bar{\tau}'$ is not a conformal density, but it is so in the dual-null gauge (43–44), which yields [18]

$$\tau + \bar{\tau}' = \delta \log \chi\bar{\chi}. \tag{83}$$

Then (τ, τ') both have conformal weight $\{1, 1\}$.

It should also be remarked that this implementation can be achieved purely in the null-tetrad formalism, without ever mentioning spinors. For instance, the weighted spin-coefficients can be expressed as (4.5.22)

$$(\kappa, \sigma, \rho, \tau) = (Dn', \delta n', \delta' n', D'n')(m)/|\chi|^2 \tag{84}$$

$$(\tau', \rho', \sigma', \kappa') = (Dn, \delta n, \delta' n, D'n)(\bar{m})/|\chi|^2. \tag{85}$$

The components of the Weyl and Ricci spinors can also be expressed in terms of the null tetrad (4.11.9–10), as can the weighted derivative operators $(\text{\TH}, \eth, \eth', \text{\TH}')$ (4.12.15) acting on tensors, and consequently all of the compacted spin-coefficient equations (4.12.32). The only spinorial remnant is the phase of χ, which does not enter tensorial expressions. In this purely tensorial view, *the desired conformal transformation is given by a simultaneous transformation of the metric and null tetrad*, (51) and (61).

V. ASYMPTOTIC REGULARITY

It is now possible to investigate the asymptotic behaviour of the space-time near both spatial and null infinity. While a detailed set of asymptotic expansions is

not derived in the present article, a leading-order analysis suffices to derive some key results.

Fixing the null coordinates $(\psi^+, \psi^-) = (\psi, \psi')/\sqrt{2}$ on $\Im$ as in §III so that it is locally a metric light cone, the conformal metric spheres are propagated into M by the vectors $(\hat{l}, \hat{l}')$, forming a preferred two-parameter family of transverse spatial surfaces, close to metric spheres, in a neighbourhood of $\Im$. It is also convenient to fix the conformal freedom by

$$(\omega, \omega') = (-\psi, \psi')/\sqrt{2} \qquad (86)$$

so that the advanced and retarded conformal factors themselves can be used as conformal null coordinates, and as expansion parameters near $(\Im^+, \Im^-)$ respectively. For expansions near both spatial and null infinity, a useful combination is

$$u = \frac{\omega \omega'}{\omega + \omega'}. \qquad (87)$$

This parameter has the properties of being linear in spatial linear combinations of (ω, ω') near i^0, with $u \sim \omega$ at $\Im^+$ and $u \sim \omega'$ at $\Im^-$, and that the constant-u hypersurfaces are hyperboloids wrapping the asymptotic light cone. The expansion parameter u is also asymptotically the inverse radius $1/r$ of the transverse surfaces, as follows. This in turn implies $\omega \sim 1/r$ at $\Im^+$ and $\omega' \sim 1/r$ at $\Im^-$.

Inspecting the form of the metric (52), a radius function can be defined by

$$r = |\chi||P_0|/|P| \qquad (88)$$

where (4.15.116)

$$P_0 = (\zeta \bar{\zeta} + 1)/\sqrt{2} \qquad (89)$$

refers to the unit sphere. The conformal radius

$$\hat{r} = |\hat{\chi}||P_0|/|\hat{P}| \qquad (90)$$

is then related by (54)–(55) as

$$\hat{r} = \omega \omega' r. \qquad (91)$$

However, for the metric spheres at $\Im$ this is just ρ of §II–III,

$$\hat{r} \approx \omega + \omega' \qquad (92)$$

and so one finds

$$u \sim 1/r. \qquad (93)$$

Here and henceforth, $a \sim b$ means $a/b \approx 1$, i.e. they have the same leading-order asymptotic behaviour at $\Im$. When considering spatial or null infinity separately, one can always use u as the sole expansion parameter, remembering that (ω, ω') count as $(u, O(1))$ at $\Im^+$, $(O(1), u)$ at $\Im^-$ and $(O(u), O(u))$ at i^0 from spatial directions. When

treating the whole of $\Im$, it is convenient to use all three expansion parameters explicitly.

Now one wishes to develop asymptotic expansions valid near both spatial and null infinity. In the usual treatment of null infinity, one can derive the asymptotic expansions from the definition of asymptotic simplicity [5]. In the current context, this is not so, since the differentiability at i^0 was deliberately left open. This issue will now be examined in terms of the spin-coefficients.

Using (63), it is straightforward to calculate the useful expressions

$$\hat{D}(\omega, \omega', u) = (-1, 0, -u^2/\omega^2)/\sqrt{2} \qquad (94)$$
$$\hat{D}'(\omega, \omega', u) = (0, 1, u^2/\omega'^2)/\sqrt{2} \qquad (95)$$
$$D(\omega, \omega', u) = (-\omega^2, 0, -u^2)/\sqrt{2} \qquad (96)$$
$$D'(\omega, \omega', u) = (0, \omega'^2, u^2)/\sqrt{2}. \qquad (97)$$

With the asymptotic gauge choice

$$\chi \approx 1 \qquad (98)$$

the conformal convergences are found as

$$\hat{\rho} = -\frac{\hat{D}\hat{r}}{\hat{r}} \sim 1/\sqrt{2}(\omega + \omega') \qquad (99)$$

$$\hat{\rho}' = -\frac{\hat{D}'\hat{r}}{\hat{r}} \sim -1/\sqrt{2}(\omega + \omega'). \qquad (100)$$

Putting these results together and using the spin-coefficient transformations (64)–(71), the physical convergences are straightforwardly calculated to have the leading-order behaviour

$$\rho \sim -u/\sqrt{2} \qquad (101)$$
$$\rho' \sim u/\sqrt{2} \qquad (102)$$

which agree with standard expressions at $\Im^+$ [5, 7, 9], but with a symmetric relative normalization of the null tetrad.

For the remaining weighted spin-coefficients, which are all conformal densities in the dual-null gauge (43–44), one can apply the generalized peeling theorem (9.7.4). In the current context, this shows that a scalar conformal density η of weight $\{k, k'\}$ satisfies $\eta = O(\omega^k)$ at $\Im^+$ and $\eta = O(\omega'^{k'})$ at $\Im^-$. This constrains the possible asymptotic behaviour at $\Im^\pm$ but does not determine it, nor the behaviour at i^0. It is tempting to conjecture the minimal asymptotic behaviour $\eta = O(\omega^k \omega'^{k'})$ for the whole of $\Im$, which turns out to be consistent with the following.

In any case, asymptotic behaviour of the conformal spin-coefficients is more precisely determined by geometrical arguments, essentially concerning the intrinsic and extrinsic curvature of a smoothly embedded metric light cone. This has already been done above for the conformal convergences (99–100). For the conformal shears, which vanish at the respective null infinity, as shown in §III or as (9.6.28), this suggests $\hat{\sigma}' = O(\omega)$ at $\Im^+$. On the

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other hand, there is no reason for $\hat{\sigma}'$ to vanish at $\Im^-$ or i^0, though it should be finite. With corresponding conditions for $\hat{\sigma}$, assuming dependence on integral powers of (ω, ω', u) uniquely yields

$$\hat{\sigma} = O(u/\omega) \tag{103}$$
$$\hat{\sigma}' = O(u/\omega'). \tag{104}$$

To be more precise, a function $f(\omega, \omega', \theta, \phi)$ is said to be *regular at $\Im$* if its limits at $\Im^+$ ($\omega \to 0$) and $\Im^-$ ($\omega' \to 0$) exist, together with the limits

$$f_0(\theta, \phi) = \lim_{\omega \to 0} f(\omega, 0, \theta, \phi) = \lim_{\omega' \to 0} f(0, \omega', \theta, \phi). \tag{105}$$

To economize on notation, $a = O(b)$ applied at $\Im$ is here and henceforth taken to mean that a/b tends to a function regular at $\Im$. This allows the limits at i^0 to depend on the angular direction, as is necessary e.g. for Ψ_2/u^3 in the Kerr solution [12].

For $(\hat{\tau}, \hat{\tau}')$, one can use the geometrical interpretation of $\hat{\tau} - \hat{\bar{\tau}}'$ as the conformal twist of the dual-null foliation, measuring the lack of commutativity of the null normal vectors (l, l') [19]. This suggests that it should be $O(\omega\omega')$. Similarly $\hat{\tau} + \hat{\bar{\tau}}'$ (83) measures transverse derivatives of the relative normalization of the null normals, suggesting similar behaviour. Then

$$\hat{\tau} = O(\omega\omega') \tag{106}$$
$$\hat{\tau}' = O(\omega\omega'). \tag{107}$$

In terms of the physical spin-coefficients, this yields

$$\sigma = O(u\omega) \tag{108}$$
$$\sigma' = O(u\omega') \tag{109}$$
$$\tau = O((\omega\omega')^2) \tag{110}$$
$$\tau' = O((\omega\omega')^2). \tag{111}$$

The practical test of these geometrically motivated conditions is whether they describe a class of space-times with desired physical properties, as considered in the next section.

Summarizing this section: leading-order behaviour of the conformal weighted spin-coefficients has been proposed according to the geometrical nature of $\Im$ as a smoothly embedded metric light cone with respect to the conformal metric. This directly determines the leading-order behaviour of the physical weighted spin-coefficients $(\kappa, \sigma, \rho, \tau, \tau', \rho', \sigma', \kappa')$. One might translate the conditions back to differentiability conditions on the metric, which would be stricter than previous suggestions [12, 13, 14], but such metric-level requirements are not particularly illuminating.

Since the asymptotic regularity conditions necessarily allow angular dependence at spatial infinity, it may sometimes be useful to expand i^0 from a point to a sphere, defined by $\omega \to 0$, $\omega' \to 0$ and parametrized by (θ, ϕ). This is in contrast to previous descriptions of i^0 as a hyperboloid [11, 12, 14], depending also on the boost direction,

or as a cylinder [15]. Here there is no detailed dependence on the boost direction ω/ω'; to use Geroch's terminology of universal versus physical structure [11], the boost dependence at i^0 is treated universally here, described using (ω, ω') in the given gauge, while the physical dependence at i^0 is purely angular, encoded in the next section in functions $(\chi_1, \rho_1, \rho_1', \sigma_1, \sigma_1')$ which are regular at $\Im$, defined as above (105) to allow only angular dependence at i^0. This seems to reflect the intuitive nature of spatial infinity as a large time-independent sphere.

VI. ENERGY

It is widely agreed that the most physically important discovery in space-time asymptotics is the Bondi-Sachs energy-loss equation [1, 2, 3], whereby the total mass-energy E_{BS} decreases at $\Im^+$ according to [5, 9, 19] $\sqrt{2}D'E_{BS} = -\oint \bar{*}|N'|^2/4\pi$, where the integral is over a sphere at $\Im^+$, $\bar{*}1$ denotes the area form of a unit sphere and the complex function N' corresponds to what Bondi dubbed the news. In more physical terminology, $|N'|^2$ is the conformal energy flux of the gravitational radiation, meaning that the physical energy flux is $|N'|^2/r^2$ near $\Im^+$, the above integral giving the integrated energy flux through a sphere near $\Im^+$. Now the news is $N' = \sigma'/u$ [19], as will be verified below. The asymptotic regularity condition (109) then implies $N' = O(\omega')$, so that it is not only finite at $\Im^+$, but decays near i^0. Transforming the Bondi-Sachs energy-loss equation to use the conformal null derivative (63) yields $\hat{D}'E_{BS} = D'E_{BS}/\omega'^2 = O(1)$. Thus the gravitational radiation decays near spatial infinity in such a way that the total energy flux, integrated over time, is finite. Moreover, the prescribed fall-off of σ' is the weakest which would achieve this result. This property has been presented first because it follows so simply from the regularity conditions, thereby verifying that the asymptotic behaviour of the shears, proposed on geometrical grounds, is exactly right on physical grounds.

It is also independent even of the existence of the total energy, for which one needs to expand some spin-coefficients further, specifically

$$\chi - 1 \sim \chi_1 u \tag{112}$$
$$-\sqrt{2}\rho/u - 1 \sim \rho_1 u \tag{113}$$
$$\sqrt{2}\rho'/u - 1 \sim \rho_1' u \tag{114}$$

where $(\chi_1, \rho_1, \rho_1')$ are regular at $\Im$. Here the proposal is that u is the natural expansion parameter near both spatial and null infinity, even though leading-order behaviour may depend on (ω, ω') independently

To study energy near $\Im$, a useful quantity is the Hawking quasi-local mass-energy [16], which can be written as [19]

$$E = \frac{A^{1/2}}{(4\pi)^{3/2}} \oint * \frac{K + \rho\rho'}{\chi\bar{\chi}} \tag{115}$$

where the integral is over a transverse surface, $*1$ denotes the area form of a transverse surface,

$$A = \oint *1 \qquad (116)$$

is the area of a transverse surface and (4.14.20)

$$K = \sigma\sigma' - \rho\rho' - \Psi_2 + \Phi_{11} + \Pi \qquad (117)$$

is the complex curvature, satisfying a complex generalization of the Gauss-Bonnet theorem (4.14.42–43):

$$\oint *\frac{K}{\chi\bar\chi} = \oint *\frac{\bar K}{\chi\bar\chi} = 2\pi(1-\gamma) \qquad (118)$$

where γ is the genus of the transverse surface, vanishing in this context. For the Schwarzschild spacetime considered in §II, one finds $\chi = (1 - 2M/r)^{1/2}$ (phase irrelevant), $\rho = -Dr/r = -(1-2M/r)/\sqrt{2}r$, $\rho' = -D'r/r = (1-2M/r)/\sqrt{2}r$, $K = 1/2r^2$ and therefore $E = M$.

The Bondi-Sachs energy [1, 2, 3] at $\Im^+$ can be expressed as [4, 5, 9, 19]

$$E_{BS} = \lim_{\omega\to 0} \frac{A^{1/2}}{(4\pi)^{3/2}} \oint *\frac{\sigma\sigma' - \Psi_2}{\chi\bar\chi}. \qquad (119)$$

In vacuum or with suitable fall-off of the matter terms Φ_{11} and Π, (117) shows that it can be written as the limit of the Hawking energy:

$$E_{BS} = \lim_{u\to 0} E. \qquad (120)$$

Since

$$*1 \;\sim\; \tilde{*}1/u^2 \qquad (121)$$
$$A \;\sim\; 4\pi/u^2 \qquad (122)$$

it is straightforward to see from the expansions (112–114) that it is finite, given specifically by

$$E_{BS} = \frac{1}{8\pi} \oint \tilde{*}(\chi_1 + \bar\chi_1 - \rho_1 - \rho_1'). \qquad (123)$$

Again one may check the Schwarzschild case: $\chi_1 = -M$, $\rho_1 = \rho_1' = -2M$, yielding $E_{BS} = M$. Note that generally *the limit of E_{BS} at spatial infinity exists uniquely* and is given by the same formula (120). This follows from the definition of functions regular at $\Im$ (105). Any such transverse surface integral of functions regular at $\Im$ has a unique limit at spatial infinity.

On the other hand, the ADM energy [10] at spatial infinity can be expressed as [11, 12, 13, 14, 20]

$$E_{ADM} = -\lim_{\omega\to 0}\lim_{\omega'\to 0} \frac{A^{1/2}}{(4\pi)^{3/2}} \oint *\frac{\Re\Psi_2}{\chi\bar\chi} \qquad (124)$$

where, in the original treatment, it was unclear whether the limit depended on the boost direction ω/ω', i.e. on the choice of spatial hypersurface. As above, this can be written

$$E_{ADM} = \lim_{\omega\to 0}\lim_{\omega'\to 0} \frac{A^{1/2}}{(4\pi)^{3/2}} \oint *\frac{K + \rho\rho' - \Re(\sigma\sigma')}{\chi\bar\chi} \qquad (125)$$

which is similar to the expression (120) for the Bondi-Sachs energy, but apparently differs by the term $\Re(\sigma\sigma')$ in the shears. The discrepancy in the two energies is found from the asymptotic regularity conditions (108–109) to be $O(\omega + \omega')$. This discrepancy is generally non-zero at null infinity ($\omega = 0$ or $\omega' = 0$), so that the ADM energy, if extended to null infinity by the same formula, would generally not agree with the Bondi-Sachs energy. However, the discrepancy vanishes at spatial infinity ($\omega = \omega' = 0$) from any direction. Thus *the ADM energy is the limit of the Bondi-Sachs energy at spatial infinity* in this context, and also the limit of the Hawking energy from any spatial or null direction:

$$E_{ADM} = \lim_{\omega\to 0}\lim_{\omega'\to 0} E = \lim_{\omega+\omega'\to 0} E_{BS}. \qquad (126)$$

This provides a remarkably simple resolution of the long-standing questions over the relation of the Bondi-Sachs and ADM energies, and the uniqueness of the latter. In the current framework, one may simply use the Bondi-Sachs energy on the entire asymptotic light cone $\Im$. The resolution explicitly rests on the additional structure at spatial infinity provided by the advanced and retarded conformal factors (ω, ω').

Returning to the energy flux of gravitational radiation: the propagation equations for the Hawking energy can be found from the compacted spin-coefficient equations (4.12.32), using $\text{Þ}*1 = -*2\rho$ and $\text{Þ}'*1 = -*2\rho'$, as [19]

$$\begin{aligned}
\text{Þ}E &= \frac{A^{1/2}}{(4\pi)^{3/2}} \Bigg[\oint *\rho\frac{K + \rho\rho' - \tau'\bar\tau' + \eth\tau' - \Phi_{11} - 3\Pi}{\chi\bar\chi} \\
&\quad + \oint *\rho'\frac{\sigma\bar\sigma + \Phi_{00}}{\chi\bar\chi} - \frac{1}{A}\oint *\rho \oint *\frac{K + \rho\rho'}{\chi\bar\chi} \Bigg] \quad(127)
\end{aligned}$$

$$\begin{aligned}
\text{Þ}'E &= \frac{A^{1/2}}{(4\pi)^{3/2}} \Bigg[\oint *\rho'\frac{K + \rho\rho' - \tau\bar\tau + \eth'\tau - \Phi_{11} - 3\Pi}{\chi\bar\chi} \\
&\quad + \oint *\rho\frac{\sigma'\bar\sigma' + \Phi_{22}}{\chi\bar\chi} - \frac{1}{A}\oint *\rho' \oint *\frac{K + \rho\rho'}{\chi\bar\chi} \Bigg] (128)
\end{aligned}$$

Rewriting these equations using the conformal spin-weighted derivatives $(\hat{\text{Þ}}, \hat\eth, \hat\eth', \hat{\text{Þ}}')$, assuming either vacuum or suitable fall-off of the matter fields, and using the asymptotic regularity conditions and expansions (108–114), it can be shown that $(\hat{D}E, \hat{D}'E)$ are $O(1)$, as expected on physical grounds. The general argument is non-trivial, subtly confirming the asymptotic regularity conditions, and uses the fact that u is constant on the transverse surfaces to cancel terms in $K + \rho\rho'$, and $\oint *\eth'\tau = 0$ (4.14.69). Fortunately the cases of most interest are $(\hat{D}E, \hat{D}'E)$ at $(\Im^-, \Im^+)$ respectively, where it is easier to see that most of the terms disappear, in the

vacuum case leaving just

$$\sqrt{2}\hat{D}E = \frac{1}{4\pi}\oint \tilde{*}|\sigma_1|^2 \quad \text{at } \Im^- \ (\omega'=0) \quad (129)$$

$$\sqrt{2}\hat{D}'E = -\frac{1}{4\pi}\oint \tilde{*}|\sigma_1'|^2 \quad \text{at } \Im^+ \ (\omega=0) \quad (130)$$

where

$$\sigma_1 = \sigma/u\omega \qquad (131)$$
$$\sigma_1' = \sigma'/u\omega' \qquad (132)$$

are the leading-order terms in the shears (108–109), regular at $\Im$. In terms of the retarded and advanced news functions

$$N = \sigma/u \qquad (133)$$
$$N' = \sigma'/u \qquad (134)$$

this implies

$$\sqrt{2}DE_{BS} = \frac{1}{4\pi}\oint \tilde{*}|N|^2 \quad \text{at } \Im^- \qquad (135)$$

$$\sqrt{2}D'E_{BS} = -\frac{1}{4\pi}\oint \tilde{*}|N'|^2 \quad \text{at } \Im^+. \qquad (136)$$

The second equation is the usual Bondi-Sachs energy-loss equation, showing that the outgoing gravitational radiation carries energy away from the system. The first equation similarly shows that ingoing gravitational radiation supplies energy to the system. The new energy propagation equations (129–130) improve on these equations by applying also in the limit at i^0. Thus the change in energy from i^0 to a section of $\Im^\pm$ is finite, as mentioned above. Physically this means that *the ingoing and outgoing gravitational radiation decays near spatial infinity such that its total energy is finite.* This property, assumed in addition to the Ashtekar-Hansen definition of asymptotic flatness [12], is known to imply the coincidence of Bondi-Sachs and ADM energies at spatial infinity [13], but here the property has been derived from the new definition of asymptotic flatness.

It is straightforward to generalize to an asymptotic energy-momentum vector

$$P = \lim_{u\to 0}\frac{A^{1/2}}{(4\pi)^{3/2}}\oint *\ell\frac{\sigma\sigma'-\Psi_2}{\chi\bar{\chi}} \qquad (137)$$

where the vector ℓ in Cartesian coordinates is (1.4.11)

$$\ell = (t,x,y,z) = \left(1, \frac{\zeta+\bar{\zeta}}{\zeta\bar{\zeta}+1}, \frac{i(\bar{\zeta}-\zeta)}{\zeta\bar{\zeta}+1}, \frac{\zeta\bar{\zeta}-1}{\zeta\bar{\zeta}+1}\right). \quad (138)$$

Then P coincides with the Bondi-Sachs energy-momentum [2, 3] at null infinity and with the ADM energy-momentum [10] at spatial infinity. The corresponding energy-momentum propagation equations are

$$\sqrt{2}\hat{D}P = \frac{1}{4\pi}\oint \tilde{*}\ell|\sigma_1|^2 \quad \text{at } \Im^- \ (\omega'=0) \quad (139)$$

$$\sqrt{2}\hat{D}'P = -\frac{1}{4\pi}\oint \tilde{*}\ell|\sigma_1'|^2 \quad \text{at } \Im^+ \ (\omega=0) \quad (140)$$

cf. [9].

Finally, it should be noted that, while the gravitational radiation at $(\Im^-, \Im^+)$ is respectively encoded as above in complex functions (σ_1, σ_1') or (N, N'), a more tensorial formulation would encode such shear terms in transverse traceless tensors, exactly as in the conventional treatment of linearized gravitational radiation.

VII. CONCLUSION

A new framework for space-time asymptotics has been proposed and developed, refining Penrose's conformal framework by introducing advanced and retarded conformal factors. This allows a relatively simple definition of asymptotic flatness at both spatial and null infinity. It has been shown how to fix the light cone at infinity so that it is locally a metric light cone. Assuming smooth embedding of the light cone, asymptotic regularity conditions have been proposed and asymptotic expansions developed. These conditions ensure that the Bondi-Sachs energy-momentum is finite and tends uniquely to the uniquely rendered ADM energy-momentum at spatial infinity, that the ingoing and outgoing gravitational radiation has the expected two modes as encoded in gravitational news functions, and that the energy-flux integrals decay at spatial infinity such that the total energy of the gravitational radiation is finite. The most basic physical properties of isolated gravitational systems have thereby been included.

Mathematically the new structure proposed for infinity is unprecedentedly rigid, allowing physical fields to have very simple behaviour at spatial infinity. Currently there are no indications that this is too simple to be physically realistic, though open issues remain. The asymptotic symmetry group presumably simplifies accordingly.

The framework, implemented in the spin-coefficient or null-tetrad formalism, is quite practical, allowing explicit calculations as for null infinity alone. While this article has presented only a leading-order analysis, asymptotic expansions can now be developed to higher orders. This would presumably allow insights into angular momentum and multipole moments of the gravitational field. A related issue is the asymptotic behaviour of the conformal curvature near spatial infinity. It may even be possible to address the long-standing question of finding a conformal form of the Einstein equations which is regular at both null and spatial infinity. In any case, it is to be hoped that the refined conformal picture will stimulate a renewal of interest in space-time asymptotics.

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